How Ideal Lattices unlocked Fully Homomorphic Encryption

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December 10, 2014

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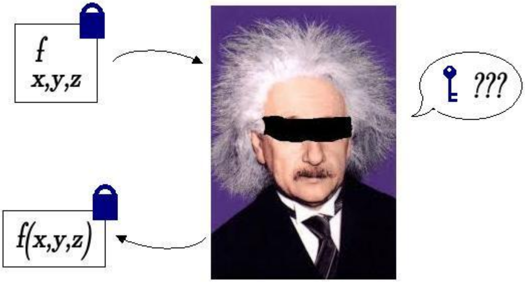
This talk

- Introduction
- Gentry's Ideal Lattices scheme
- Further advances, others schemes and open problems

Fully Homomorphic Encryption

Question : "Is it possible to compute blindfolded?"



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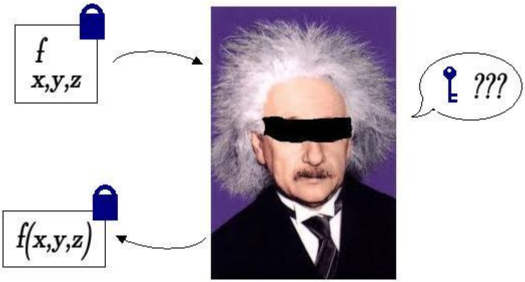

Example : A public-key cryptosystem $\mathcal{E}$ verifying : $\forall a, b \in \mathcal{P}(\mathcal{E})$,

$$a + b = \mathrm{D}_{\mathcal{E}}(\mathrm{E}_{\mathcal{E}}(a) + \mathrm{E}_{\mathcal{E}}(b)),$$
$$a \times b = \mathrm{D}_{\mathcal{E}}(\mathrm{E}_{\mathcal{E}}(a) \times \mathrm{E}_{\mathcal{E}}(b)).$$

Formal definition

Def. 1 : A **homomorphic scheme** is a public-key scheme $\mathcal{E}$ with four PPT algorithms :

- KeyGen: $\lambda \mapsto (\mathtt{sk}, \mathtt{pk})$;
- Enc: $(m, \mathtt{pk}) \mapsto c$;
- Dec: $(c, \mathtt{sk}) \mapsto m$;
- Eval:$(C, c_1, \ldots, c_n, \mathtt{pk}) \mapsto m$.

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Def. 2 : A homomorphic scheme is *correct* for a set of circuits $\mathcal{C}$ if, for every circuit in $\mathcal{C}$,

$$\psi \leftarrow \mathtt{Eval}(C, \psi_1, \ldots, \psi_n, \mathtt{pk}) \;\Rightarrow\; \mathtt{Dec}(\psi, \mathtt{sk}) = C(\pi_1, \ldots, \pi_n)$$

where $\psi_i = \mathtt{Enc}(\pi_i, \mathtt{pk}), i = 1, \ldots, n$.

A **Fully Homomorphic Scheme** is a homomorphic scheme that is correct for all circuits.

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- Let I be an ideal of some ring R,
- $m \in R$ the message.

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Encryption : $\texttt{Enc}(m) = m + xI$ for some $x \in R$.

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And strong security properties.

Ideals + lattices = Ideal lattices

Let $R = \mathbb{Z}[X]/(X^n + 1)$ where n is a power of 2, and consider the mapping $\alpha : R \to \mathbb{Z}^n$,

$$\alpha(v_0 + v_1 X + \cdots + v_{n-1} X^{n-1}) = (v_0, v_1, \cdots, v_{n-1})$$

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I.e., the columns of $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Operations in an ideal lattice

Let L be an ideal lattice with base $\mathbf{B}_L = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Define

$$P(\mathbf{B}_L) = \left\{ \sum_{i \leq n} x_i \mathbf{b}_i \in \mathbb{R}^n \ ; \ x_i \in [-1/2, 1/2) \right\}.$$

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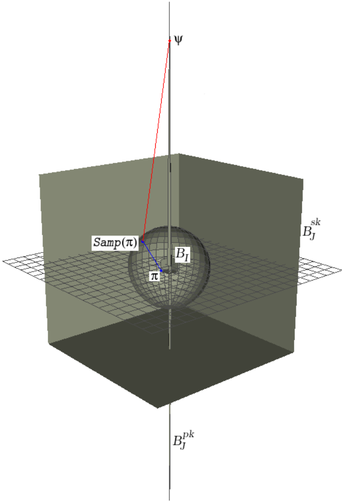
- Addition in $\mathbb{Z}^n$: $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y}$
- Product in $\mathbb{Z}^n$: $(\mathbf{x}, \mathbf{y}) \mapsto \alpha(\mathbf{x}(X) \times \mathbf{y}(X))$

Gentry's solution

- Let J be an ideal lattice, generated by two bases $\mathbf{B}_J^{\mathrm{sk}}, \mathbf{B}_J^{\mathrm{pk}}$.
- $\mathcal{P} \subseteq \{0,1\}^n$, $\mathtt{pk} = \{\mathbf{B}_J^{\mathrm{pk}}\}$, $\mathtt{sk} = \{\mathbf{B}_J^{\mathrm{sk}}\}$
- Let $\mathtt{Samp}(\pi)$ be a (bounded) random algorithm that samples from $\pi + 2\mathbb{Z}^n$.

Gentry's solution



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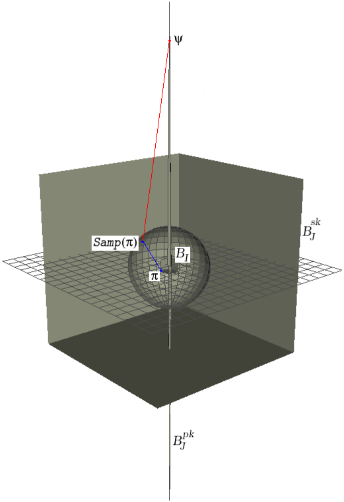

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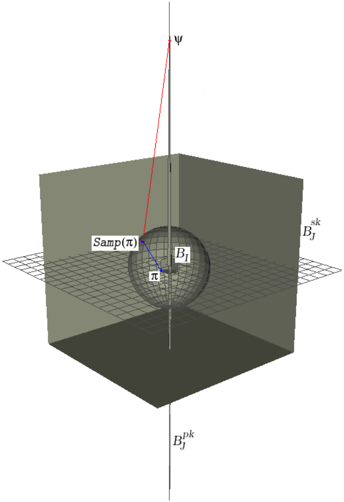

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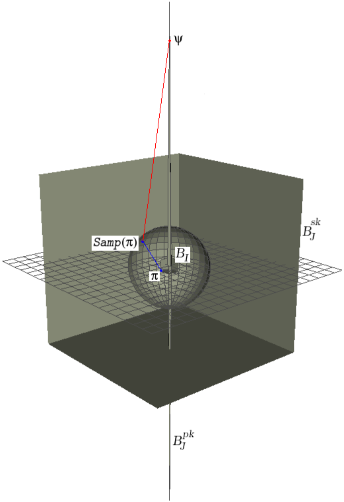

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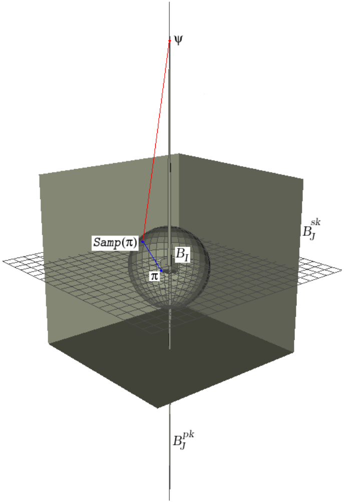

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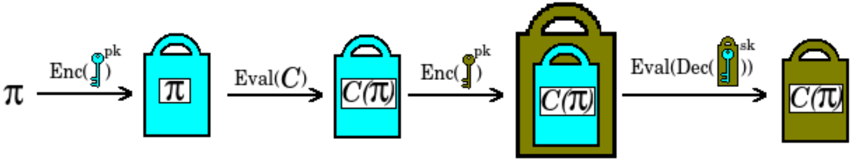
Theorem : $d_{\max} = \log\log ||\vec{v}_{\mathtt{sk}}||_2 - \log\log(\sqrt{n} \cdot l_{\mathtt{Samp}})$

Ground-breaking idea

Bootstrapping : Capability of refreshing a high-noise message.



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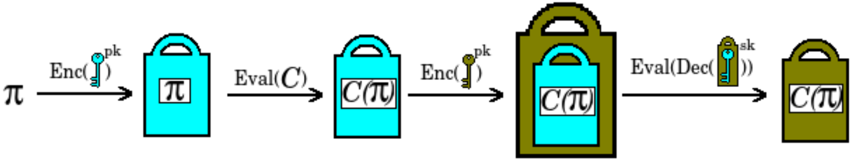

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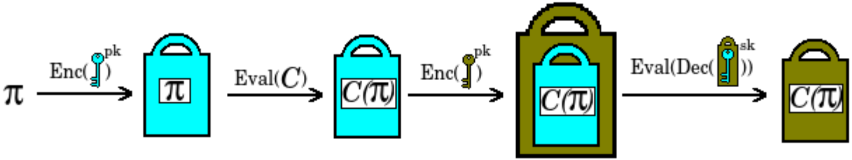

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- Introduces "circular security" issues.

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Gentry reduces the degree of the decryption circuit and achieves bootstrapping.

New security issues

Circular security : Is it safe to send Key-Dependent messages ? If so; is this provable ?

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The Sparse Subset Sum Vector Problem : Given an upper bound for θ, distinguish between

$$\{\vec{t}_1, \ldots, \vec{t}_\Theta\} \subset^R \mathbb{Q}^n \text{ and } \{\vec{t}_1, \ldots, \vec{t}_\Theta \in \mathbb{Q}^n; \sum_{i \in S} \vec{t}_i = 0\}.$$

Other FHE schemes

van Dijk, Gentry, Halevi, Vaikuntanathan.– A FHE scheme over $\mathbb{Z}$.

Brakerski, Vaikuntanathan.— (i) FHE from LWE (ii) FHE with proved circular security

Multikey FHE

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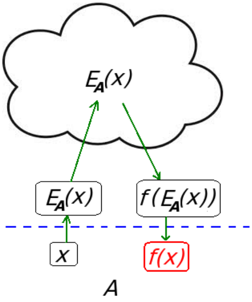
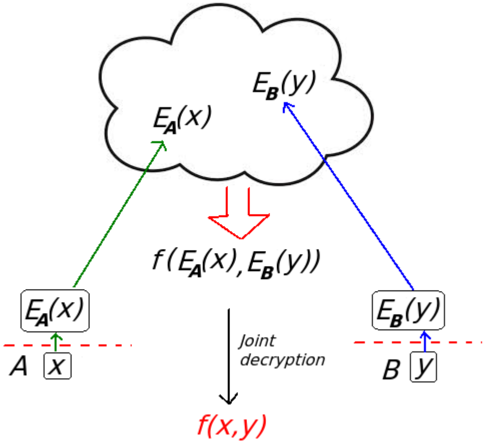


Figure : Single Key FHE scenario

MPC on the cloud



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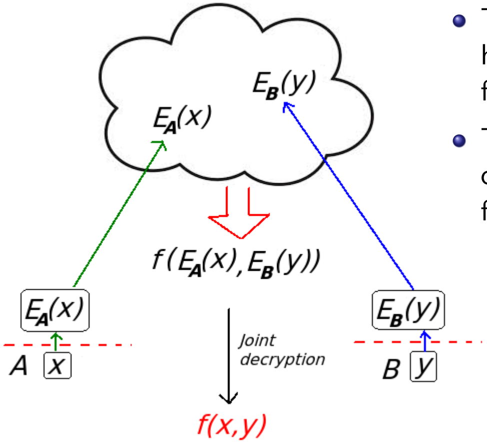

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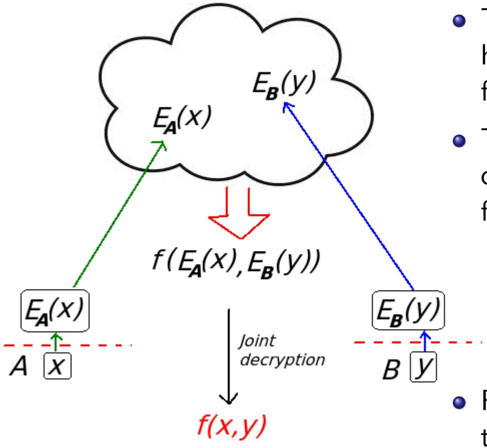

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- Reduction of general MPC to a particular MPC !

Attribute-Based and Identity-Based FHE scheme

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- or into a (hierarchical) identity based scheme.

Hierarchical encryption

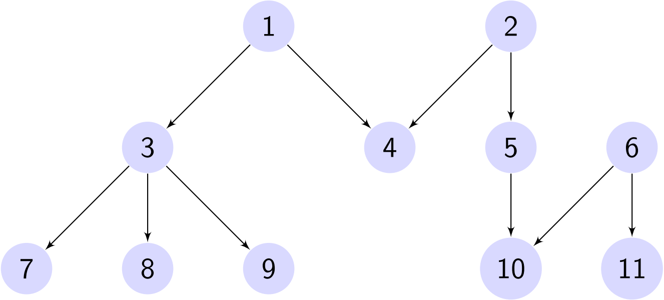

Figure : A polytree.

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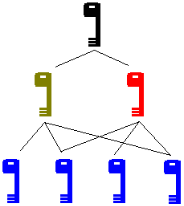

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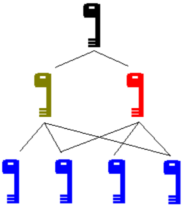

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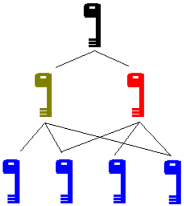

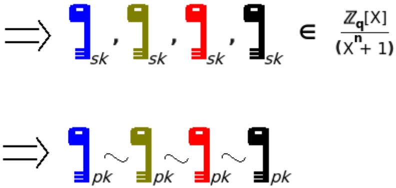

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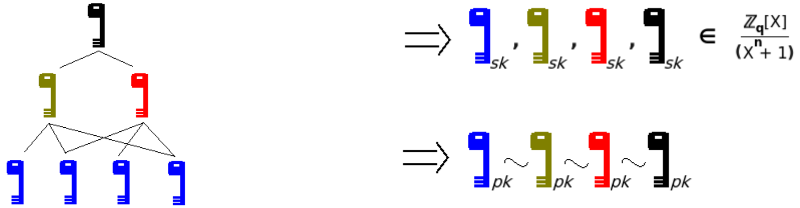

- Changes can be done in the tree in real time
- Two distant users can collaborate regardless of the authority level

(Work in progress...)

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- Is it possible to exploit the "graph structure" on ciphertexts via $C + E(0)$ or $C \times E(1)$?

Thank you!