Ideal lattices in number theory and in cryptology

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Definition of homomorphic encryption

We assume the usual setting in cryptography

- Secret Key S_k.
- Public key P_k (or another secret key).
- Encryption function $\mathrm{Enc}(m, P_k)$.
- Decryption function $\mathrm{Dec}(c, S_k)$ such that $\mathrm{Dec}(\mathrm{Enc}(m, P_k), S_k) = m$.

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- Homomorphic for f of limited complexity : somewhat homomorphic.
- Homomorphic for any f : fully homomorphic.

Motivation of homomorphic encryption

Versailles

Hospital

Far far away

Server farm

Somewhere in between

evil NSA, semi-evil GCHQ, good DGSE

Motivation of homomorphic encryption

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Sends encrypted update

$\mathsf{Enc}(x, S_k)$

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Updates $c' = f(c, x)$

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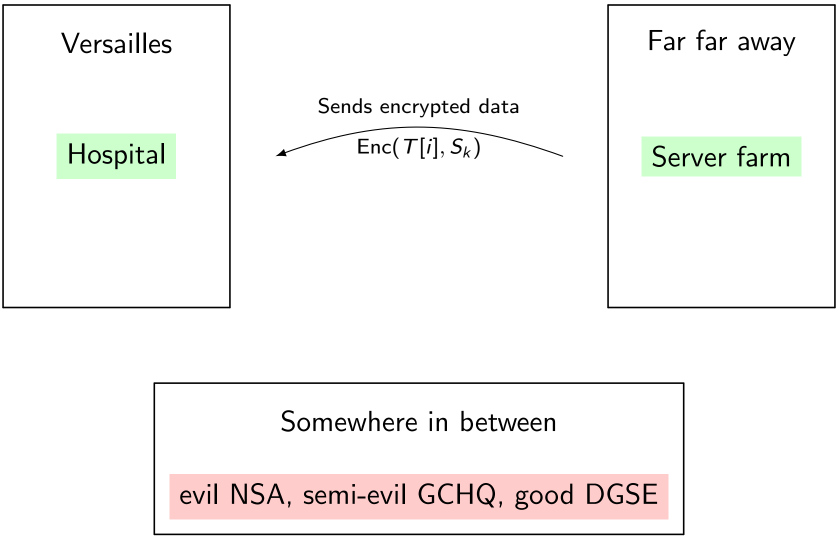


Quantum Computing

Peter Shor

Polynomial time algorithms for

- Integer factoring.
- Solving the Discrete Logarithm Problem.

Hard problem 1 : (R)-LWE

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Encryption with (R)-LWE

Let $m \in \{0, 1\}$ be a message and s a secret.

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1. Draw a and e at random.
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- (R)-LWE assumption : it is hard to distinguish many $(a, a \cdot s + e)$ from randomness.
- It reduces to finding short vectors in (ideal)-lattices.

Hard problem 2 : short-PIP

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Let $m \in \{0, 1\}$ and a prime ideal $\mathfrak{p} = (g)$ where g is small and secret.

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Encryption with short-PIP

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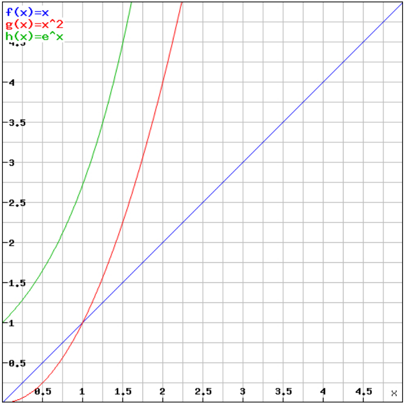
1. Draw a at random.
2. Send $C = m + 2 * a \bmod \mathfrak{p}$.

- short-PIP assumption : hard to find a short generator of an ideal $\mathfrak{p}$.
- Not sure what it reduces to, but it can be solved by solutions to the shortest vector problem.

Hardness of a problem

We quantify it by the function Time $= f(\text{input size})$.



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For $a \in [0, 1]$, $L_S(a, b)$ is between exponential and polynomial in S

$$L_S(0, b) = S^b,$$
$$L_S(1, b) = (e^S)^b.$$

Number fields

Let $K/\mathbb{Q}$ of degree n, $\mathcal{O}_K$ its **ring of integers**, $\sigma : K \rightarrow \mathbb{C}$ its embeddings.

$$U := \mathcal{O}_K^* = \mu \times \langle \epsilon_1 \rangle \times \cdots \times \langle \epsilon_r \rangle,$$

where $r = n_1 + n_2$ with n_1 is the number of real embeddings of K, and n_2 the paire of complex ones.

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The **fractional ideals** are of the form $\mathfrak{a} = \frac{1}{d} I$ where I is an ideal of $\mathcal{O}_K$.

$$\mathrm{Cl}(\mathcal{O}_K) := \{\text{Invertible fractional ideals}\}/\{\text{Principal fractional ideal}\}.$$

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- We have $[\mathfrak{a}] = [\mathfrak{b}] \in \mathrm{Cl}(\mathcal{O}_K)$ if $\mathfrak{a} = (\alpha)\mathfrak{b}$ for some $\alpha \in K$.
- In cryptography $K = \mathbb{Q}[X]/\Phi_N(X) = \mathbb{Q}(\xi_N)$ for $\xi_N := e^{\frac{2i\pi}{N}}$.

Computing the class group and the unit group

We assume we are given K, its n_1 real embeddings, its n_2 pairs of complex embeddings and $\mathcal{O}_K = \sum_i \mathbb{Z}\alpha_i$.

Output of the algorithm

We compute $d_1, \cdots, d_k \in \mathbb{Z}$ and $\gamma_1, \cdots, \gamma_r$ such that

- $\mathrm{Cl}(\mathcal{O}) = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}$.
- $U = \mu \times \langle\gamma_1\rangle \times \cdots \times \langle\gamma_r\rangle$

where $r := r_1 + r_2 - 1$ and μ is the set of roots of unity.

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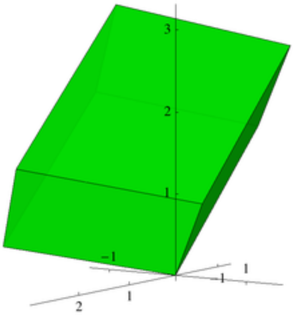
We show how to use U and $\mathrm{Cl}(\mathcal{O}_K)$ to solve the PIP

$$\text{Given } \mathfrak{a}, \quad \text{find } g \text{ such that } \mathfrak{a} = (g)\mathcal{O}_K$$

Input size



Let $(\omega_i)_{i\leq n}$ the integral basis of $\mathcal{O}$ (given as input), its fundamental volume is

$$\Delta = \det \begin{pmatrix} \sigma_1(\omega_1) & \dots & \sigma_1(\omega_n) \\ \vdots & \ddots & \vdots \\ \sigma_n(\omega_1) & \dots & \sigma_n(\omega_n) \end{pmatrix}^2$$

Class group computation

Factor base

Let $\mathcal{B} = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_N\}$ be a set of ideals whose classes generate $\mathrm{Cl}(\mathcal{O})$.

We consider the surjective morphism

$$\begin{array}{ccccc} \mathbb{Z}^N & \xrightarrow{\varphi} & \mathcal{I} & \xrightarrow{\pi} & \mathrm{Cl}(\mathcal{O}) \\ (e_1, \ldots, e_N) & \longrightarrow & \prod_i \mathfrak{p}_i^{e_i} & \longrightarrow & \prod_i [\mathfrak{p}_i]^{e_i} \end{array}$$

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We therefore deduce $\mathrm{Cl}(\mathcal{O})$ from the lattice of the $(e_1, \cdots, e_N)$ such that

$$\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_N^{e_N} = (\alpha) = 1 \in \mathrm{Cl}(\mathcal{O}) \text{ for some } \alpha \in \mathcal{O}.$$

From relations to the unit group

Let $M \in \mathbb{Z}^{N \times N'}$ be a relation matrix. That is,

$$\forall i, \mathfrak{p}_1^{m_{i,1}} \cdots \mathfrak{p}_N^{m_{i,N}} = (\alpha_i) = 1 \in \mathrm{Cl}(\mathcal{O}).$$

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Property

Let $X = (x_1, \cdots, x'_N)$ be such that $XM = 0$. Then,

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We use the following strategy

- We derive units from elements in $\ker(M)$.
- We construct a minimal generating set for U by induction.

From relations to a generator of $\mathfrak{a}$

Assume the rows of $M \in \mathbb{Z}^{l \times k}$ generate all the relations of the form

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Problem : most likely g is not short.

Short generators of ideals

Let $\mathfrak{a} \subseteq \mathcal{O}$ a principal ideal and $U = \mu \times \langle \varepsilon_1 \rangle \times \cdots \times \langle \varepsilon_r \rangle$ the unit group.

- We know how to compute generators for U and a generator of $\mathfrak{a}$.
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When $r = 1$, then we find $e \in \mathbb{Z}$ such that $\log |\alpha| - e \log |\varepsilon|$ has the desired size.

- Let $\mathrm{Log}(x) := (\log |x|_1, \cdots, \log |x|_r) \in \mathbb{R}^r$.
- We want $\| \mathrm{Log}(\alpha) + \sum_i e_i \mathrm{Log}(\varepsilon_i) \|_2$ small.

In arbitrary dimension, we want to solve the closest vector problem.

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Buchmann 90

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- Ideal class group/Unit group.
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In classes of **fixed degree** number fields.

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B. - Fieker 14

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Quantum Computing

Finding the periods of a function

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Hidden Subgroup Problem (HSP)

We look for a discret subgroup $G \subseteq \mathbb{R}^m$ such that there is f satisfying

- $f(x+g) = f(x)$ for all $g \in G$.
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There exists a quantum algorithm to solve the HSP in polynomial time.

Reducing factorization to HSP

Suppose we want to factor $N = pq$. Let a coprime with N and

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The function f "hides" the subgroup $G = r\mathbb{Z} \subseteq \mathbb{Z}$.

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- The function f "hides" the subgroup $G = \mathbb{Z} \times (x\mathbb{Z}) \subseteq \mathbb{Z} \times \mathbb{Z}$.
- Finding x reduces to solving the HSP with f.

Reducing the ideal class group and unit group to HSP

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Remark

- We have a well-defined function only when the degree is fixed.
- We can derive solutions to the PIP easily.

Reducing the unit group to HSP (large degree)

We have n_1 real embeddings and n_2 complex embeddings of K.

- We have the natural inclusion $K \hookrightarrow \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}$.

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Let S be a set of prime ideals of $\mathcal{O}_K$, an S-unit is an $x \in K$ satisfying

$$\exists (e_1, .., e_{|S|}) \in \mathbb{Z}^{|S|},\ x \cdot \mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_{|S|}^{e_{|S|}}.$$

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- We reduce the class group and the PIP to the computation of U_S.

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Let x an S-unit and $v_i \in \mathbb{Z}$ such that $v_i = v_{\mathfrak{p}_i}$, then

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Let $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_N\}$ be a set of ideals whose classes generate $\mathrm{Cl}(\mathcal{O})$.

$$\text{Under GRH, } |S| \leq 12 \log^2 |\Delta|.$$

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Which gives us $\mathrm{Cl}(\mathcal{O}) \simeq \mathbb{Z}^N / \ker(\pi \circ \varphi)$.

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- We rewrite $\mathfrak{a}$ as a power-product of the $\mathfrak{p}_j$ given by the vector b.
- We solve the linear system $XH = b$.

The daunting question

Given $\mathfrak{a} = (g)$ in $\mathcal{O}_K$ the ring of integers of K. we know how to compute

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The daunting question : How do we find the small ones ?